Mathematics Internal Assessment
Higher Level

Topic: Limits of the ratio of consecutive terms in Second Order Recurrence Relations

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1. Introduction and Background Information

The Fibonacci sequence is a series where the n^{th} term is the sum of the previous two terms, $n-1^{th}$ and $n-2^{th}$. Moreover, the first term, f_0, is assumed to be 0 and the second term, f_1, to be 1. Mathematically, one can represent this series as:

$$f_n = f_{n-1} + f_{n-2} \qquad for\ n \geq 2, n \in \mathbb{Z}^+, f_0 = 0, f_1 = 1$$

where f_n is the n^{th} term of the series.

Such kind of a relationship, where a term f_n depends on its two immediate predecessors, f_{n-1} and f_{n-2}, is called a *second order linear homogeneous recurrence relation*.

1.1 Rationale

When my mathematics teacher introduced us to converging series while doing limits as a part of Option 3: Calculus, he gave us a question where we had to find $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$ where $f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi^{-1})^n)$. I solved it, just to find out that the answer was the golden ratio (ϕ) itself. This got me intrigued. Later, I went home and tried doing the same with Lucas Sequence, a series very similar to Fibonacci Sequence, where the only difference is in the values of L_0 and L_1, which are predefined. The answer turned out to be ϕ again. Upon doing further exploration with the help of books such as Ralph P. Grimaldi's Discrete and Combinatorial Mathematics, and by finding patterns in many other similar series, I found out that $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$, where $G_n = xG_{n-1} + yG_{n-2}$ is always $\frac{x+\sqrt{x^2+4y}}{2}$ assuming $x^2 + 4y \geq 0$ irrespective of the initial values, G_0 and G_1.

In this investigation, I aim to explore second order recurrence relations and arrive at this particular result.

To understand second order relation, I would like to start by briefing first order relations using geometric progression.

1.2 Geometric Progression

The geometric progression is an infinite sequence of numbers, for instance 2, 10, 50, 250, 1250, …, where the division of each term, other than the first, by its immediate predecessor is a constant, called the common ratio. For the sequence given above, the common ratio, $r = \frac{a_1}{a_0} = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_{n+1}}{a_n}$ is 5. Hence, the progression we have is $a_n = 5a_{n-1}$ for $n > 0, n \in \mathbb{Z}^+$. This is a first order recurrence relationship because any arbitrary term a_n is dependent on its immediate predecessor a_{n-1}. Generally, a first order recurrence relation is expressed in the form of $a_n = ra_{n-1}$. However, this expression can be used for infinitely such relations, which is why is it important to define the value of the first term, a_0. Once a_0 is defined and r is found, we will obtain a specific series.

It is important to note that there are limitations to the values of r: $r \in \mathbb{R}, r \neq 0$. If $r = 0$, then the series will not be in geometric progression anymore.

Continuing with the example already in hand, we have

$$a_0 = 2$$

$$a_1 = 5(a_0) = 5(2) = 10$$

$$a_2 = 5(a_1) = 5\big(5(2)\big) = 5^2(2)$$

$$a_3 = 5(a_2) = 5\big(5^2(2)\big) = 5^3(2)$$

Therefore, $$a_n = 2(5^n)$$

In general, any geometric progression can be expressed as

$$a_n = a_0 r^n \text{ for } n \geq 0, n \in \mathbb{Z}^+ ---------------(1)$$

Notice that both the expressions $a_n = a_0 r^n$ and $a_n = r a_{n-1}$ will result in the same series for a particular value of a_0 .

The expression (1) is required as I will be using this in the second order recurrence relationship.

Now, that we have a fair idea of first order recurrence relations, I would like to define what a homogeneous relation is.

1.3 Homogenous Relation

Let $k \in \mathbb{Z}^+$ and $C_0 (\neq 0), C_1, C_2, \dots C_k (\neq 0)$ be real numbers. If a_n, for $n \geq 0$, is a discrete function, then

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = F(n), \; n \geq k$$

is a linear recursion (with constant coefficient) of order k. When $F(n) = 0$ for all $n \geq 0$, the relation is called *homogeneous*. If $F(n) \neq 0$, it is called a *nonhomogeneous* relation[1].

In this exploration, we will only consider homogeneous relations. Nonhomogeneous relations are beyond the scope of the IA.

For a second order relation, therefore, $k = 2$ and,

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0$$

According to the equation (1), $a_n = a_0 r^n$. So, the second order relation is now

$$C_0 a_0 r^n + C_1 a_0 r^{n-1} + C_0 a_0 r^{n-2} = 0$$

$$\Longrightarrow a_0 (C_0 r^n + C_1 r^{n-1} + C_2 r^{n-2}) = 0$$

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[1] Grimaldi, Ralph P. *Discrete and Combinatorial Mathematics: an Applied Introduction*. Pearson Addison Wesley, 2014.

Hence,

$$\Rightarrow C_0r^n + C_1r^{n-1} + C_2r^{n-2} = 0$$

As it is second order, $n = 2$ and $r \neq 0$, so we arrive at

$$C_0r^2 + C_1r + C_2 = 0 ------------------(2)$$

Let the roots of this equation be r_1 and r_2. We will then split it into three cases. One thing we assume in these cases is that the roots are *linearly independent*, which means that one root is not the multiple of other, for all $n \in \mathbb{N}$.

Case 1: Distinct Real Roots.

If the two roots r_1 and r_2 are real and distinct, the general solution for the given equation will be given as

$$a_n = c_1r_1^n + c_2r_2^n$$

where c_1 and c_2 are arbitrary constants[2].

Case 2: Repetitive Real Root.

If the two roots r_1 and r_2 are real and equal, the general solution for the given equation will be given as

$$a_n = c_1r^n + c_2nr^n$$

where c_1 and c_2 are again arbitrary constant

Case 3: Complex Roots.

If the two roots r_1 and r_2 are complex, the general solution for the given equation will be given as

$$a_n = k_1r^n \cos n\alpha + k_2r^n \sin n\alpha$$

where $r_1 = re^{\alpha i}$, $r_2 = re^{-\alpha i}$, $k_1 = c_1 + c_2$ and $k_2 = (c_1 - c_2)i$

2. Exploration

2.1 Conjecture 1: Explicit Representation of Fibonacci Sequence

Coming back to the Fibonacci sequence where $f_n = f_{n-1} + f_{n-2}$ (for $n \geq 2, n \in \mathbb{N}$) with the initial conditions gives as $f_0 = 0$ and $f_1 = 1$

$$f_n = f_{n-1} + f_{n-2}$$

$$\Rightarrow f_n - f_{n-1} - f_{n-2} = 0$$

[2] Lerma, Miguel A. "Recurrence Relations." 30 Oct. 2003.

By using Equation (2), we get

$$r^2 - r - 1 = 0$$

Using the general formula to solve a quadratic equation:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$

The value $\frac{1+\sqrt{5}}{2}$ is denoted as ϕ (the golden ratio) and, therefore, $\frac{1-\sqrt{5}}{2}$ is $-\phi^{-1}$

If $b^2 - 4ac = 0$, the formula for *Repetitive Real roots* (case 2) is to be used.

If $b^2 - 4ac < 0$, the formula for *Complex roots* (case 3) is to be used.

As the two roots are real and distinct, the general formula for *distinct roots* is used as follows

$$f_n = c_1\phi^n + c_2(-\phi^{-1})^n$$

We know the values of f_n at $n = 0$ and $n = 1$ as they are predefined for the Fibonacci Sequence. Therefore,

$$f_0 = c_1 + c_2 = 0$$

$$f_1 = c_1\phi + c_2(-\phi^{-1}) = 1$$

By solving the equations simultaneously, we arrive at c_1 to be $\frac{1}{\sqrt{5}}$ and c_2 to be $\frac{-1}{\sqrt{5}}$.

Substituting these values in the equation of L_n, we get the expression

$$f_n = \frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}(-\phi^{-1})^n$$

$$\Rightarrow f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi^{-1})^n)$$

2.2 Proof of First Conjecture

Now that I have found a conjecture for the explicit representation of the Fibonacci sequence, the next thing to do is to prove it. I would be using the *second principle of mathematical induction* to prove the conjecture. It states that if, for any statement involving a positive integer, n, the following are true:

1. The statement holds for $n = 1$
2. Whenever the statement holds for all $n \leq k$, it must also hold for $n = k + 1$

then the statement holds for all positive integers, n.

Before starting the proof, it is important to realize that $1+\frac{1}{\phi} = \phi$ and $1-\phi = -\frac{1}{\phi}$

For $n = k$,

$$f_k = \frac{1}{\sqrt{5}}(\phi^k - (-\phi^{-1})^k)$$

For $n = k-1$,

$$f_{k-1} = \frac{1}{\sqrt{5}}(\phi^{k-1} - (-\phi^{-1})^{k-1})$$

$$= \frac{1}{\sqrt{5}}\left(\phi^k.\frac{1}{\phi} - \left((-\frac{1}{\phi})^k\right).(-\phi)\right)$$

We have to prove that it is true for $n = k+1$

$$f_{n+1} = f_n + f_{n-1}$$

$$\Rightarrow LHS = \frac{1}{\sqrt{5}}(\phi^k - (-\phi^{-1})^k) + \frac{1}{\sqrt{5}}\left(\phi^k.\frac{1}{\phi} - \left((-\frac{1}{\phi})^k\right).(-\phi)\right)$$

$$= \frac{1}{\sqrt{5}}\left(\phi^k.(1+\frac{1}{\phi}) - \left((-\frac{1}{\phi})^k\right).(1-\phi)\right)$$

$$= \frac{1}{\sqrt{5}}\left(\phi^k.(\phi) - \left((-\frac{1}{\phi})^k\right).\left(-\frac{1}{\phi}\right)\right)$$

$$= \frac{1}{\sqrt{5}}\left(\phi^{k+1} - \left((-\frac{1}{\phi})^{k+1}\right)\right)$$

$$= RHS \text{ for } n = k+1$$

Hence, the conjecture

$$f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi^{-1})^n)$$

is now proved using the second principle of mathematical induction. The final expression is commonly known as Binet's Formula.

2.3 Finding $\lim_{n\to\infty}\frac{G_{n+1}}{G_n}$ for Fibonacci Series

The next part is to find $\lim_{n\to\infty}\frac{f_{n+1}}{f_n}$, where $f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi^{-1})^n)$

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{5}}(\phi^{n+1} - (-\phi^{-1})^{n+1})}{\frac{1}{\sqrt{5}}(\phi^n - (-\phi^{-1})^n)}$$

$$= \lim_{n\to\infty} \frac{\left(\phi^{n+1} - \left(\frac{-1}{\phi}\right)^{n+1}\right)}{\left(\phi^{n} - \left(\frac{-1}{\phi}\right)^{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\frac{\left(\phi^{n+1} - \left(\frac{-1}{\phi}\right)^{n+1}\right)}{\phi^{n+1}}}{\frac{\left(\phi^{n} - \left(\frac{-1}{\phi}\right)^{n}\right)}{\phi^{n+1}}}$$

$$= \lim_{n\to\infty} \frac{\left(1 - \frac{(-1)^{n+1}}{\phi^{2n+2}}\right)}{\left(\frac{1}{\phi} - \frac{(-1)^{n}}{\phi^{2n+1}}\right)}$$

$$= \frac{(1-0)}{\left(\frac{1}{\phi} - 0\right)}$$

$$= \phi$$

Therefore, $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \phi$, where $f_n = \frac{1}{\sqrt{5}}(\phi^n - (-\phi^{-1})^n)$

2.4 Finding $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$ for Lucas Series

Now, I will try to find the same thing using a different sequence called the Lucas Sequence. The latter is similar to the former and the only difference between them is the set of initial conditions. The Lucas Sequence is defined as

$$L_n = L_{n-1} + L_{n-2} \quad for\ n \geq 2, n \in \mathbb{Z}^+, f_0 = 2, f_1 = 1$$

$$\Longrightarrow L_n - L_{n-1} - L_{n-2} = 0$$

By using Equation (2) again, we get

$$r^2 - r - 1 = 0$$

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$

As the two roots are real and distinct, we use the general formula for *distinct roots* as follows

$$L_n = c_1\phi^n + c_2(-\phi^{-1})^n$$

As we know the initial conditions for Lucas Sequence as well, we get

$$L_0 = c_1 + c_2 = 2$$

$$L_1 = c_1\phi + c_2(-\phi^{-1}) = 1$$

By using simultaneous equations, we arrive at c_1 to be 1 and c_2 to be 1.

Substituting these values in the equation of L_n, we get the expression

$$\Rightarrow L_n = (\phi^n - (-\phi^{-1})^n)$$

Next, we need to prove that $\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \phi$, where $L_n = (\phi^n - (-\phi^{-1})^n)$

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \lim_{n\to\infty} \frac{(\phi^{n+1} - (-\phi^{-1})^{n+1})}{(\phi^n - (-\phi^{-1})^n)}$$

$$= \lim_{n\to\infty} \frac{\left(\phi^{n+1} - \left(\frac{-1}{\phi}\right)^{n+1}\right)}{\left(\phi^n - \left(\frac{-1}{\phi}\right)^n\right)}$$

$$= \lim_{n\to\infty} \frac{\left(1 - \frac{(-1)^{n+1}}{\phi^{2n+2}}\right)}{\left(\frac{1}{\phi} - \frac{(-1)^n}{\phi^{2n+1}}\right)}$$

$$= \frac{(1-0)}{\left(\frac{1}{\phi} - 0\right)}$$

$$= \phi$$

It is important to realize that even though the initial values of the Lucas Series, L_0 and L_1, are different from those of the Fibonacci Series, f_0 and f_1, their final answer tends to be the same. This is because those values give a constant that gets cancelled while finding $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$ for both Fibonacci and Lucas series.

2.5 Proof of First Generalization

Therefore, we can generalize it by saying that for any sequence where $G_n = G_{n-1} + G_{n-2}$ and $n \geq 2$, the $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$ will always be equal to ϕ irrespective of the value of G_0 and G_1. To prove this, we can define G_n as

$$G_n = G_{n-1} + G_{n-2} \qquad for\ n \geq 2, n \in \mathbb{Z}^+, G_0 = a, G_1 = b$$

$$\Longrightarrow G_n - G_{n-1} - G_{n-2} = 0$$

By using Equation (2) again, we get

$$r^2 - r - 1 = 0$$

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$

$$G_n = c_1\phi^n + c_2(-\phi^{-1})^n$$

We know that $G_0 = a$ and $G_1 = b$

$$G_0 = c_1 + c_2 = a - - - - - - - - - - - - - - - - - - - (3)$$

$$G_1 = c_1\phi + c_2(-\phi^{-1}) = b - - - - - - - - - - - - - - - - - (4)$$

Substituting ϕ as $\frac{1+\sqrt{5}}{2}$ in equation (4) we get,

$$c_1(\frac{1+\sqrt{5}}{2}) + c_2\left(-(\frac{1+\sqrt{5}}{2})^{-1}\right) = b$$

From equation (3), we have $c_2 = a - c_1$

$$\Rightarrow c_1(\frac{1+\sqrt{5}}{2}) + (a - c_1)\left(\frac{-2}{1+\sqrt{5}}\right) = b$$

$$\Rightarrow \frac{c_1(1+\sqrt{5})^2 + (a-c_1)(-4)}{2(1+\sqrt{5})} = b$$

$$\Rightarrow c_1(6+2\sqrt{5}) + 4c_1 - 4a = 2b(1+\sqrt{5})$$

$$\Rightarrow 2c_1(5+\sqrt{5}) - 4a = 2b(1+\sqrt{5})$$

$$\Rightarrow c_1(5+\sqrt{5}) - 2a = b(1+\sqrt{5})$$

$$\Rightarrow c_1 = \frac{b(1+\sqrt{5}) + 2a}{5+\sqrt{5}}$$

As $c_2 = a - c_1$

$$c_2 = a - \frac{b(1+\sqrt{5}) + 2a}{5+\sqrt{5}}$$

$$\Longrightarrow c_2 = \frac{a(5+\sqrt{5}) - b(1+\sqrt{5}) - 2a}{5+\sqrt{5}}$$

Therefore,

$$G_n = \left(\frac{b(1+\sqrt{5}) + 2a}{5+\sqrt{5}}\right)\phi^n - (\frac{a(5+\sqrt{5}) - b(1+\sqrt{5}) - 2a}{5+\sqrt{5}})(-\phi^{-1})^n$$

Next, we need to find $\lim_{n\to\infty}\frac{G_{n+1}}{G_n}$

$$\lim_{n\to\infty}\frac{G_{n+1}}{G_n} = \lim_{n\to\infty}\frac{\left(\frac{b(1+\sqrt{5}) + 2a}{5+\sqrt{5}}\right)\phi^{n+1} - (\frac{a(5+\sqrt{5}) - b(1+\sqrt{5}) - 2a}{5+\sqrt{5}})(-\phi^{-1})^{n+1}}{\left(\frac{b(1+\sqrt{5}) + 2a}{5+\sqrt{5}}\right)\phi^{n} - (\frac{a(5+\sqrt{5}) - b(1+\sqrt{5}) - 2a}{5+\sqrt{5}})(-\phi^{-1})^{n}}$$

$$= \lim_{n\to\infty} \frac{\left(b\left(1+\sqrt{5}\right)+2a\right)\phi^{n+1} - \left(a\left(5+\sqrt{5}\right)-b\left(1+\sqrt{5}\right)-2a\right)(-\phi^{-1})^{n+1}}{\left(b\left(1+\sqrt{5}\right)+2a\right)\phi^{n} - \left(a\left(5+\sqrt{5}\right)-b\left(1+\sqrt{5}\right)-2a\right)(-\phi^{-1})^{n}}$$

$$= \lim_{n\to\infty} \frac{\dfrac{\left(b\left(1+\sqrt{5}\right)+2a\right)\phi^{n+1} - \left(a\left(5+\sqrt{5}\right)-b\left(1+\sqrt{5}\right)-2a\right)(-\phi^{-1})^{n+1}}{\phi^{n+1}}}{\dfrac{\left(b\left(1+\sqrt{5}\right)+2a\right)\phi^{n} - \left(a\left(5+\sqrt{5}\right)-b\left(1+\sqrt{5}\right)-2a\right)(-\phi^{-1})^{n}}{\phi^{n+1}}}$$

$$= \lim_{n\to\infty} \frac{\left(b\left(1+\sqrt{5}\right)+2a\right) - \left(a\left(5+\sqrt{5}\right)-b\left(1+\sqrt{5}\right)-2a\right)\dfrac{(-1)^{n+1}}{\phi^{2n+2}}}{\dfrac{\left(b\left(1+\sqrt{5}\right)+2a\right)}{\phi} - \left(a\left(5+\sqrt{5}\right)-b\left(1+\sqrt{5}\right)-2a\right)\dfrac{(-1)^{n}}{\phi^{2n+1}}}$$

$$= \frac{\left(b\left(1+\sqrt{5}\right)+2a\right)}{\dfrac{\left(b\left(1+\sqrt{5}\right)+2a\right)}{\phi}}$$

$$= \phi$$

Hence, we have proved that for every sequence defined by $G_n = G_{n-1} + G_{n-2}$ and $n \geq 2$, the $\lim\limits_{n\to\infty} \frac{G_{n+1}}{G_n}$ will always be equal to ϕ, irrespective of the values of G_0 and G_1.

2.6 Proof of Second Generalization

Till now, we have only explored sequences defined in the form of $G_n = G_{n-1} + G_{n-2}$. Next, we shall deal with sequences in the form of $G_n = xG_{n-1} + yG_{n-2}$, where $n \geq 2,\ x, y \in \mathbb{R}, G_0 = a, G_1 = b$.

$$G_n = xG_{n-1} + yG_{n-2}$$

$$\Longrightarrow G_n - xG_{n-1} - yG_{n-2} = 0$$

By applying equation (2) we get

$$r^2 - xr - y = 0$$

Using the general formula to solve a quadratic equation:

$$r = \frac{-(-x) \pm \sqrt{x^2 - 4(1)(-y)}}{2(1)}$$

$$\Longrightarrow r = \frac{x \pm \sqrt{x^2 + 4y}}{2}$$

Therefore,

$$r_1 = \frac{x+\sqrt{x^2+4y}}{2} \text{ and } r_2 = \frac{x-\sqrt{x^2+4y}}{2}$$

Substituting these values back to the initial equation, we get

$$G_n = c_1 r_1^n + c_2 r_2^n$$

It is important to note that $r_2 = -{r_1}^{-1}$. Hence,

$$G_n = c_1 r_1^n + c_2(-{r_1}^{-1})^n$$

We know that irrespective of the value of G_0 and G_1, the final value of $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$ will remain the same. As $G_0 = a$ and $G_1 = b$,

$$G_0 = c_1 + c_2 = a$$

$$G_1 = c_1 r_1 + c_2(-{r_1}^{-1}) = b$$

Solving the simultaneous equation will result in the value of c_1 and c_2, but for simplicity, I will not solve it and take them as c_1 and c_2 only.

Our goal is to find $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$, hence, it is the next step.

$$\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \lim_{n\to\infty} \frac{c_1 r_1^{n+1} + c_2(-{r_1}^{-1})^{n+1}}{c_1 r_1^n + c_2(-{r_1}^{-1})^n}$$

$$= \lim_{n\to\infty} \frac{\frac{c_1 r_1^{n+1} + c_2(-{r_1}^{-1})^{n+1}}{r_1^{n+1}}}{\frac{c_1 r_1^n + c_2(-{r_1}^{-1})^n}{r_1^{n+1}}}$$

$$= \lim_{n\to\infty} \frac{c_1 + c_2 \frac{(-1)^{n+1}}{(r_1)^{2n+2}}}{\frac{c_1}{r_1} + c_2 \frac{(-1)^n}{(r_1)^{2n+1}}}$$

$$= \frac{c_1 + 0}{\frac{c_1}{r_1} + 0}$$

$$= r_1$$

$$= \frac{x + \sqrt{x^2 + 4y}}{2}$$

However, there is a limitation here. $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$ will only exist when $x^2 + 4y \geq 0$. If $x^2 + 4y < 0$, $\frac{x+\sqrt{x^2+4y}}{2}$ would result in a complex root.

Hence, for any series defined in the form of $G_n = xG_{n-1} + yG_{n-2}$, where $n \geq 2$, $x, y \in \mathbb{R}$, $G_0 = a, G_1 = b$ the value of $\lim_{n\to\infty} \frac{G_{n+1}}{G_n} = \frac{x+\sqrt{x^2+4y}}{2}$, assuming that $x^2 + 4y \geq 0$. This is also the root, r_1, of the equation $r^2 - xr - y = 0$ where $r_1 > r_2$ and $r_1$and r_2 are the two distinct roots.

3. Scope for Further Work:

a) One way to move forward with this idea is to explore if a similar result can be found out for nonhomogeneous relations.
b) A second way to further explore in this area is to test whether similar patterns arrive at higher order recurrence relations.
c) When any general relation $G_n = xG_{n-1} + yG_{n-2}$ is there, an assumption is being made that n is greater than or equal to 2 $(n \geq 2)$. A way to move forward with this is to find out what happens when $n < 2$. Does the hypothesis still hold true? Is there any other pattern?

4. Conclusion:

In this exploration, I arrived at an explicit way to represent any series and proved that $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$, where G is a series and its n^{th} term is defined by $G_n = xG_{n-1} + yG_{n-2}$, is always equal to $\frac{x+\sqrt{x^2+4y}}{2}$, provided that $x^2 + 4y \geq 0$, hence, meeting with the aim of this exploration. While doing this, I also proved that the initial values, G_0 and G_1 do not affect the value of $\lim_{n\to\infty} \frac{G_{n+1}}{G_n}$.

5. Reflection:

While doing this exploration, I learnt about Homogeneous Recurrence Relations and I got very engrossed in the broader topic of Recurrence Relations. As this topic was not done in school, I initially had a lot of difficulty understanding the math behind it, but with the help of my teacher I was able to do so. I used the book 'Discrete and Combinatorial Mathematics' by Ralph P. Grimaldi to further solve questions based on it. After I got comfortable with the it, I even tried to seek for patterns in higher order recurrence relations, but was unable to find one.

The most important thing that I have learnt in to not trust every source, especially forums, where people discuss questions based on the topic. Some of the explanations are completely flawed, because of which I too had difficulty understanding the concept.

6. Bibliography:

1. Grimaldi, Ralph P. *Discrete and Combinatorial Mathematics: an Applied Introduction*. Pearson Addison Wesley, 2014.
2. Lerma, Miguel A. “Recurrence Relations.” 30 Oct. 2003.